Conclave meeting of the RTG 2491 "Fourier Analysis and Spectral Theory"

Organized by
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Introduction by the Organizers

The RTG 2491 "Fourier Analysis and Spectral Theory" is the home of mathematical research and education of doctoral students. The research focuses around a common methodology: we apply techniques from spectral theory, in particular Fourier analysis and harmonic analysis. We do this in a variety of areas, from number theory to to differential equations and mathematical physics.

Mathematical research tends to be quite specialized and sophisticated. As a consequence, it is often not easy to communicate what we are actually working at, and why this is exciting us so much —as typically it does.

But it is a matter of fact that the truly exciting progress in most cases arises at the borderline of areas and from the combination of ideas from different fields. This is true for scientific advances in general, and also for the specific area of mathematics, and also in a subfield. If you have a chance to talk to research mathematicians, be it doctoral students or retired professors at the end of their career: often they will tell you that the decisive idea for a proof or a theorem arose from a chance encounter with someone with different background and perspective, or from a presentation in a rather different field.

Consequence: we need to be experts in our specialty, but at the same time we need to learn to understand mathematics outside of our direct area of research. And we need to use the opportunities to communicate with others: excite them of our specific subfield, and become excited by what they are doing.

To foster this, we got together in a small hamlet in the hills south of Göttingen, cutting ourselves even off some of the modern communication channels and used the opportunity of intense interaction.

As part of this we set ourselves the goal that each of us would explain to a peer his research, but then the non-expert would present this to the world.

The fruit of this you have in your hands: 15 short portraits of a research program; where Alice is explaining us the research agenda of Bob, while Bob describes what Alice is doing. Now the public is invited to read the outcome of the effort and to get engaged in even more discussions. Have fun!

A snapshot of our RTG Group

List of Reports

Reports

On the existence of solutions for Diophantine equations

CHRISTIAN BERNERT
Communicated by *Léo Bénard*

Christian works in the field of analytic number theory. Let (E) be a Diophantine equation – that is, a polynomial equation involving a polynomial of finitely many variables with integral coefficients, such as

$$17X^3 + 23XYZ + 1013W^4 + 1 = 0.$$

Christian seeks to bound the minimal size of an integral solution of (E).

Motivation for this work comes from the following question, known as *Hilbert's tenth problem*:

Does there exist an algorithm to decide in finite time whether a Diophantine equation has an integral solution or not?

This problem was answered negatively in 1970 by Matiyasevich, continuing previous work by Robinson, Davis and Putnam. This theorem has, however, not discouraged the attempts to find such algorithms, at least for more restricted families of equations.

- Considering the simple case of a linear equation of two variables

(1.1) $$ax + by = c$$

  for integers a, b and c, it is not difficult to see that (1.1) has an integral solution if and only if c is a multiple of the greatest common divisor of a and b. It turns out that similar criteria exist for linear equations with any number of variables.

- Although degree two is more involved, it is again possible to decide if a given equation has a solution.

  Theorem 1.2 (Local-global principle). *A Diophantine equation of degree two has a non-trivial integral solution if and only if it has non-trivial solutions over $\mathbb{R}$ and modulo p^k for all primes p, for all natural numbers k.*

  By non-trivial solution, we simply want to discard the case of a homogeneous polynomial having trivially the solution $(0, \ldots, 0)$.

  Note that given such an equation (E), it suffices to check that it has a non-trivial solution modulo p^k for a finite number of primes powers. Hence this theorem provides indeed a finite algorithm.

- In degree 3, the local-global principle no longer holds. The equation $3X^3 + 4Y^3 + 5Z^3 = 0$ is known to provide a counterexample.

Take a (cubic) equation (E): $C(\underline{x}) = 0$, with $\underline{x} = (x_1, \ldots, x_n)$. Now the Hardy–Littlewood circle method comes into play. It is a machinery that may give an asymptotic formula for

$$N_P(E) = \#\{\underline{x} \mid C(\underline{x}) = 0, |x|_\infty \leq P\},$$

where P is a positive real number. When the machinery works, the asymptotic term has the form

(1.3) $$N_P(E) \sim \kappa P^{n-3} + O(P^{n/2})$$

with some constant κ.

In the case of a cubic form (homogeneous degree 3 equation), this technique is powerful enough to provide existence of non-trivial solutions as soon as the number of variables exceeds 14, and this bound is expected to be optimal.

Assuming that (E) admits non-trivial solutions, Christian works in finding an explicit bound $P = P(E)$ such that $N_P(E)$ is effectively non-zero. Note that in an algorithmic perspective, the existence of such a bound says that one can search for integral solutions of (E) in a finite set.

Let M be the maximum of the coefficients involved in the equation (E). Christian showed that for a cubic form, if $n \geq 14$, then $P = M^{150000}$ is enough for large M.

Less is known in the non-homogeneous case. Christian focuses on cubic equations (E) whose *rank* is big enough. The rank (or h-invariant) of an equation (E): $C(\underline{x}) = 0$ is the minimal number h such that C can be written as

$$C = L_1 Q_1 + \ldots L_h Q_h$$

for L_i at most linear, Q_i at most quadratic.

Intuitively, the smaller this invariant is, the more degenerate is the equation. For a generic cubic C, its rank equals the number of variables n. The important feature is that if h is big enough, then the Hardy–Littlewood machinery works.

In a work in progress, Christian shows that for $h \geq 14$, there exists a k such that for $P = M^k$, the number $N_P(E)$ of solutions smaller that P is non-zero.

The idea of the proof is to control the error term in the asymptotic formula (1.3). A control on the constant κ is given when solving the equation (E) modulo prime powers, namely κ can be expressed as the product of the normalized numbers of solutions mod p^m and of real solutions. Those numbers are bounded, at least in the case of a smooth cubic, by Deligne's work on Weil's conjectures. In the singular case, more work is required, and this is the actual content of Christian's efforts.

On representation theory for Lie algebroids

Geoffrey-Desmond Busche
Communicated by *Jialong Deng*

Several mathematicians working in Göttingen have made huge impact in the development of Sophus Lie's theory regarding the structures known as Lie algebras and Lie groups today. For instance, the friendship of Felix Klein and Lie in their younger years influenced each other's mathematical ideas, the effect of Hilbert's fifth question and von Neumann's result on on it, as well as Hermann Weyl's work on the representations of Lie algebras and Lie groups and his classic book *The Classical Groups*, among many others. The research program regarding the representation theory for Lie algebroids in the RTG 2491 will continue progressing in Lie's theory.

According to Weinstein, though groups are indeed sufficient to characterize homogeneous structures, there are plenty of objects which exhibit what we clearly recognize as symmetry, but which admit few or no nontrivial automorphisms. It turns out that the symmetry, and hence much of the structure, of such objects can be characterized algebraically if we use groupoids and not just groups. Groupoids were first introduced (and named) by H. Brandt in 1926. Namely a groupoid is a small category in which every morphism is an isomorphism, i.e. invertible. More precisely:

Definition 2.1 (Groupoid). A groupoid G consists of two sets: a set of objects G_0 and a set of arrows G_1 together with five structure maps s, t, u, i, c, namely

(1) the source and target maps $s, t\colon G_1 \to G_0$ (in category theoretic terms, domain and codomain of arrows),

(2) the unit inclusion $u\colon G_0 \to G_1$, mapping an object to its identity morphism,

(3) the composition of arrows $c\colon G_2 \to G_1$, defined on the fibered product $G_2 = G_{1s}\times_t G_1 = \{(h,g) \in G_1\times G_1 : s(h) = t(g)\}$ and written $c(h,g) = hg$ (the set G_2 is the set of pairs of composable arrows),

(4) the inversion of arrows $i\colon G_1 \to G_1$, written $i(g) = g^{-1}$,

which fulfil the following conditions:

(1) Compatibility of source and target with composition: For all $(h,g) \in G_2$ we have $s(hg) = s(g)$ and $t(hg) = t(h)$.

(2) Associativity: For all $k, h, g \in G_1$ such that $s(k) = t(h)$ and $s(h) = t(g)$ we have $(kh)g = k(hg)$ (note that these compositions make sense because of property 1).

(3) Compatibility of the unit with source, target and composition in the sense that

    (a) for all $x \in G_0$ we have $su(x) = x = tu(x)$ and

    (b) for all $g \in G_1$ we have $[ut(g)]g = g = g[us(g)]$.

(4) Compatibilities of the inverse: For all $g \in G_1$ we have $s(g^{-1}) = t(g)$ and $t(g^{-1}) = s(g)$ as well as $g^{-1}g = us(g)$ and $gg^{-1} = ut(g)$.

It is easy to see that any group is also a groupoid, giving the most basic example. Furthermore, every set can be regarded as a groupoid that only has identity morphisms. Literature contains many other examples.

Roughly, Lie groups are groups enriched with a smooth structure. Similarly, a Lie groupoid is a groupoid combined with smooth structures.

Definition 2.2 (Lie groupoid). A Lie groupoid is a groupoid G for which, in addition, G_0 and G_1 are smooth manifolds, and the structure maps s, t, u, i and c are smooth. Furthermore, s and $t\colon G_1 \to G_0$ are required to be submersions.

In category theoretic language, a Lie groupoid is a groupoid whose sets of objects and of morphisms are both manifolds, whose source and target operations are submersions, and in which all the category operations (source and target, composition, and identity-assigning map) are smooth. Any Lie group gives a Lie groupoid with one object, and conversely. So, the theory of Lie groupoids includes the theory of Lie groups. Given any manifold M, there is a Lie groupoid called the pair groupoid, with M as the manifold of objects, and precisely one morphism from any object to any other. In this Lie groupoid the manifold of morphisms is thus $M \times M$.

Definition 2.3 (Lie algebroid). A Lie algebroid is a vector bundle $A \to M$ together with a vector bundle morphism $\rho\colon A \to TM$ (called anchor map) and a Lie bracket $[-,-]$. on the space of sections of A, satisfying the Leibniz rule $[X, fY] = f[X, Y] + \rho(X)(f)Y$ for all smooth sections X and Y of A, $f \in C^\infty(M)$. Here, $\rho(X)(f)$ is the Lie derivative of f along the vector field $\rho(X)$.

Every Lie algebra is a Lie algebroid over the one-point manifold. Every involutive subbundle of the tangent bundle — that is, one whose sections are closed under the Lie bracket — also defines a Lie algebroid. The representation theory of Lie algebras and Lie groups is a way to study Lie's theory and plays an important role in theoretic physics. Therefore, one can hope that the theory of Lie algebroid and Lie groupoid representations has similarly meaningful physical applications. Geoffrey-Desmond Busche is making progress in that direction.

Like the infinitesimal approximation to a Lie group is a Lie algebra, the infinitesimal approximation to a Lie groupoid is a Lie algebroid. So to every Lie groupoid, a Lie algebroid is associated. Unfortunately, going back from a Lie algebroid to a Lie groupoid is not always possible, but every Lie algebroid gives a so called stacky Lie groupoid at least. Geoffrey-Desmond Busche's goal is to differentiate Lie groupoid representations to Lie algebroid representations and integrate them back. The the exponential map (similar to the exponential map in Lie group theory) will be used among other tools to connect both sides.

Why you cannot have bulges everywhere

Jialong Deng

Communicated by *Geoffrey-Desmond Busche*

Looking at the world surrounding us, at things like an apple, a salad bowl or a mug of coffee in our hands, we usually have an intuitive understanding of curvature. We know how to distinguish convex from concave optical lenses and bulges from dents. It is hence no wonder that curvature is a classical subject of mathematics since the early days of modern research. Famous in many different areas, Carl Friedrich Gauss studied curvature during the early 19th century and came up with an innovative notion that was later named after him. The Gaussian curvature describes a (two-dimensional) surface, embedded in Euclidean space, by approximating the intersections of the surface and a normal plane with a tangent circle. Gauss' work was both inspired by and used in the measurement of the kingdom of Hanover and has since proven its practical value countless times. Purely mathematical questions quickly arose as well. Gauss himself conjectured that his notion of curvature was independent of the surrounding space and proved this to be true in his *Theorema egregium* with much fewer tools than today's mathematicians can access.

From that point, a logical and fruitful generalization is to omit the space surrounding the object of interest completely. Today, we do not restrict ourselves to the investigation of two-dimensional embedded surfaces. Instead we can use the theory of Riemannian manifolds (which means manifolds with a scalar product on each tangent space, depending smoothly on the base point). Since the days of Gauss, several different descriptions of curvature have been invented, each with its respective appeals. Of particular interest to Jialong Deng is the *scalar* curvature, which measures how the volume of a ball of given radius deviates from the respective volume of a ball in Euclidean space of the same size. More concretely, this works as follows.

Given a Riemannian manifold (M, g) of dimension m, we have an induced volume form $\omega = \mathrm{d}g \in \Omega^m(M)$, which in return yields a measure μ on M by $\mu(U) = \int_U \omega$ for open subsets U of M. In addition, there is the geodesic distance function $d \colon M \times M \to \mathbb{R}_{\geq 0}$, attaching to a pair of points the minimal length of a straight line connecting them. Given a point $p \in M$ and a radius $r \in \mathbb{R}_{>0}$, we can naturally define the ball of radius r around p as $B_p(r) = \{x \in M : d(p, x) < r\}$. This ball has a volume $\mathrm{vol}_g(B_p(r)) = \mu(B_p(r))$ which depends smoothly on both the point and the radius.

An m-dimensional Euclidean ball of radius r has its usual volume $\mathrm{vol}_{\mathbb{R}^m}(B(r)) = c(m) \cdot r^m$, where $c(m) \in \mathbb{R}_{>0}$ is a constant depending only on m; for example, $c(2) = 2\pi$ and $c(3) = \frac{4\pi}{3}$. The scalar curvature $\mathrm{sc}(p)$ of (M, g) at the point p is

then defined so that

$$\text{vol}_g(B_p(r)) = \text{vol}_{\mathbb{R}^m}(B(r)) \cdot \left(1 - \frac{\text{sc}(p)}{6(m+2)} r^2 + \mathcal{O}(r^4)\right)$$

for sufficiently small radii r. Differentiating with respect to r two times transforms this to the following explicit equation:

$$\text{sc}(p) = -3(n+2) \frac{\text{d}^2}{\text{d}r^2}\bigg|_{r=0} \frac{\text{vol}_g(B_p(r))}{\text{vol}_{\mathbb{R}^m}(B(r))}$$

In dimension 2, the scalar curvature is just the double of the Gaussian curvature. Hence it extends the classical theory to higher dimensions.

The scalar curvature depends on the chosen Riemannian metric, which is more information than the topology and smooth structure of the manifold. It turns out, however, that the topology alone creates constraints for the curvature functions that may be realized by a Riemannian metric on it. Namely, there are smooth manifolds which do not admit any Riemannian metric with positive scalar curvature everywhere; in this sense, they cannot be bulged at every point. The simplest example of this is the 2-dimensional torus $S^1 \times S^1$.

The question which functions can be realized as the scalar curvature of some metric on a given manifold is known as the *prescribed curvature problem*. Kazdan and Warner have progressed in this area by proving a theorem (see [1], Theorem 4.35) that separates smooth manifolds of dimension three or higher into three classes: On manifolds of the first class, every smooth function can be realized as scalar curvature. The second class admits curvatures which are constantly zero or negative at some point, and the third one only allows curvatures which are negative somewhere. By this theorem, finding manifolds which admit a positive scalar curvature becomes equal to finding the manifolds where every curvature function can be realized.

This task is currently pursued by Jialong Deng. He progresses using the theory of minimal surfaces and index theory, including the K-theory of C^*-algebras.

Just as in the days of Gauss, the investigation of curvature still has practical applications today. While harder to grasp with our intuition than the surface of the earth, the Einstein field equation is a tool used by physicists to describe our universe in general relativity. One important term in this equation is the scalar curvature of space time. Chances are high that future researchers will continue to use the theory of scalar curvature for a long time.

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Topological T-duality

Tom Dove

Communicated by *Eske Ewert*

It is a constant goal of physics to develop models of the universe that accurately describe our world. In particular, physicists are looking for a unified theory that explains all four fundamental forces: gravity, electromagnetism and the strong and weak forces. String theory is a candidate for such a theory. Within string theory there is a relationship between space-time models called *T-duality*. If two models are T-dual to each other, they are physically equivalent, even if they appear to be very different. Tom is studying topological T-duality, which describes the topology that underlies this relationship.

In its topological formulation, a physical model is described by a *pair* (E, h) over a space B, consisting of a principal $\mathbb{T}^n$-bundle $E \to B$ together with a *twist h* which represents a cohomology class in $H^3(E; \mathbb{Z})$. This twist is called the *H-flux* and can take the form of, for example, a gerbe or a closed differential 3-form.

A *gerbe* is an object that geometrically represents a class in $H^3(E; \mathbb{Z})$ in the same way line bundles represent classes in $H^2(E; \mathbb{Z})$. For the latter, taking the first Chern class of a line bundle allows to identify the isomorphism classes of line bundles with $H^2(E; \mathbb{Z})$. Similarly, $H^3(E; \mathbb{Z})$ can be understood as isomorphism classes of gerbes.

Given two pairs (E, h) and $(\widehat{E}, \widehat{h})$ over B, one can form the fibre product $E \times_B \widehat{E}$. Loosely speaking, the T-dual relation is an isomorphism between the H-fluxes h and $\widehat{h}$ when pulled back to the fibre product. In addition, this isomorphism must satisfy a Poincaré bundle condition.

The simplest cases are principal $\mathbb{T}^1$-bundles. In this situation, it turns out that the T-dual of a given pair (E, h) is the bundle whose Chern class is the pushforward of $[h]$ along the projection $E \to B$. The dual H-flux is chosen in such a way that the relation is symmetric. In particular, every pair has a T-dual which is unique up to isomorphism.

However, for $\mathbb{T}^n$-bundles with $n \geq 2$ not every pair admits a T-dual and the T-dual need not be unique.

An important aspect of T-duality is that it induces isomorphisms in certain *twisted cohomology theories*. These isomorphisms are in fact predicted by physicists. For a smooth manifold M and a closed 3-form ω *twisted de Rham cohomology* is constructed as follows. The de Rham differential d is twisted by adding ω to it:

$d_\omega = d + \omega$. Denote the resulting 2-periodic cohomology groups by $H^\bullet(M, \omega)$. If (E, h) and $(\widehat{E}, \widehat{h})$ are T-dual to each other, there is an isomorphism in twisted de Rham cohomology with a degree shift: $H^\bullet(E, h) \cong H^{\bullet+1}(\widehat{E}, \widehat{h})$.

Another example is *twisted K-theory*. The H-flux h yields a class in $H^3(E; \mathbb{Z})$, which is isomorphic to $H^2(E; \mathbb{T}^1)$. If the class of h is torsion, this viewpoint can be used to define twisted vector bundles over E. An ordinary k-dimensional vector bundle is determined by its transition functions $\varphi_{ij} \colon U_i \cap U_j \to \mathrm{Gl}(k)$ between local trivializations. The transition functions satisfy a cocycle condition. One can construct twisted vector bundles by replacing the cocycle condition with $\varphi_{ij}\varphi_{jk} = \varphi_{ik}h_{ijk}$ for a torsion class $[h] \in H^2(E; \mathbb{T}^1)$. Building on these twisted vector bundles, the twisted K-group $K^0(E, h)$ is defined. T-duality again induces isomorphisms between the respective twisted K-groups with a degree shift.

Tom will look into the question of what can be said in more singular physical situations. These can be modelled by torus actions that are no longer free. So far, there have been several approaches to this in the case of non-free circle actions. Tom will compare these to find the right axiomatic definition of pairs and T-duality in this context. Furthermore, he will investigate under which criteria the T-dual exists and when it is unique. A related question is how many T-duals a given pair has.

After this, Tom will construct the isomorphisms between twisted cohomology groups that arise from the T-duality relation in this broader framework. A further direction will be to understand in a systematic way which twisted cohomology theories admit T-duality isomorphisms.

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Filtered manifolds and non-elliptic operators

Eske Ewert

Communicated by *Tom Dove*
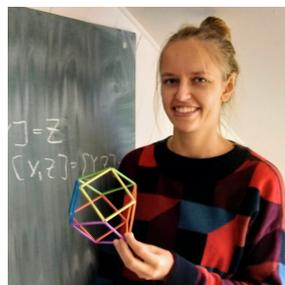
Fredholm operators are operators whose kernel and cokernel are of finite dimension. Named in honour of Erik Ivar Fredholm, these operators were first used to study integral equations, and have been widely thought about since. An important class of Fredholm operators are elliptic operators on compact manifolds; these are the subject of the famous Atiyah-Singer Index Theorem, which demonstrates a powerful connection between analysis and topology by proving that the analytic index and topological index of an elliptic operator are equal. However, there are operators that are Fredholm, but not elliptic. In Eske Ewert's recently completed PhD thesis, *Index theory and groupoids for filtered manifolds*, she develops a mathematical framework to study such operators.

How does one test for ellipticity in the classical setting? Consider a differential operator P on $\mathbb{R}^n$. This has the form

$$P = \sum_{|\alpha| \leq m} c_\alpha \partial^\alpha,$$

for a collection of smooth functions $c_\alpha \colon \mathbb{R}^n \to \mathbb{C}$. To check if this is an elliptic operator, one considers the highest order part,

$$\sigma(P) = \sum_{|\alpha| = m} c_\alpha \partial^\alpha.$$

For a fixed $x \in \mathbb{R}^n$, this is a constant coefficient operator on $\mathbb{R}^n$. Then one applies the Fourier transform to obtain the *principal symbol* of P,

$$p(x, \xi) = \sum_{|\alpha| = m} c_\alpha(x)(-i\xi)^\alpha.$$

If the principal symbol is invertible for all x, then P is elliptic.

Now for a non-elliptic Fredholm operator. Consider the 3-dimensional Heisenberg group, whose Lie algebra is generated by X, Y, Z satisfying $[X, Y] = Z$ and $[X, Z] = [Y, Z] = 0$. The operator

$$-X^2 - Y^2 + i\mu Z, \quad \mu \in \mathbb{C} \setminus 2\mathbb{Z} + 1,$$

on a quotient of the Heisenberg group by an integer lattice is Fredholm and non-elliptic. This is because Z is not present in the highest order part of the operator and thus is not in the principal symbol either. A remedy for this is to assign new orders to the variables: one asserts that X and Y will now have order 1 and Z will

now have order 2. Then Z belongs to the highest order part, and we get a different principal symbol. To make the idea of assigning different orders to vector fields precise, Eske considers *filtered manifolds*, an idea which goes back to the 1970s and has recently been used in noncommutative geometry. A filtered manifold is a smooth manifold paired with a filtration of its tangent bundle by smooth subbundles, and satisfying a certain "integrability" condition for its sections. The assigning of orders to vector fields then corresponds to which subbundles the vector fields belong to.

When working in the framework of filtered manifolds, the highest order part of an operator is interpreted as a right-invariant differential operator on some nilpotent Lie group coming from the filtration of the filtered manifold. In the classical setting above, the highest order part is an operator on $\mathbb{R}^n$ and so one can comfortably make use of the Fourier transform. On a filtered manifold, the Fourier transform is replaced by a study of the irreducible representations of the underlying Lie group. The notion that certain operators can be better understood when having their highest order part acting on nilpotent Lie groups goes back to the work of Folland, Rothschild and Stein in the 1970s, and has since been used several times in the study of pseudo-differential operators.

With a new notion of order, one can then investigate new notions of principal bundles and ellipticity. In the smooth manifold setting, there is an approach to pseudo-differential operators using the tangent groupoid. This groupoid was defined by Alain Connes and used in his proof of the Atiyah-Singer Index Theorem. For a manifold M, the tangent groupoid is the groupoid with arrows

$$TM \times \{0\} \cup (M \times M) \times (0, \infty).$$

For filtered manifolds, one also has a tangent groupoid; in this case the tangent bundle is replaced by a bundle of Lie groups (precisely those being acted on by the highest order part). From here one obtains a noncommutative algebra of symbols, which is the target of a principal symbol map from the space of compact operators. Then, one simply defines an operator to be elliptic if its symbol is invertible in this algebra.

In her thesis, Eske uses filtered manifolds and their groupoids as a framework for studying non-elliptic Fredholm operators. This approach allows Eske to ascertain a number of their properties and head towards an index theory for non-elliptic operators. In her future work, Eske hopes to use these methods to define and investigate new pseudo-differential calculi.

In search for ideals in the world of Lie algebroids

Rosa Marchesini

Communicated by *Jérémy Mougel*

In this article, we will follow a scientist in his quest to find ideals. This quest takes place in the fabulous world of Lie algebroids. A Lie algebroid is a triple $(A, [\cdot, \cdot], \rho)$ consisting of a vector bundle $A \to M$ over a smooth manifold M. This vector bundle is equipped with a Lie bracket $[\cdot, \cdot]$ on $\Gamma(A)$, the space of section of A. The vector bundle A is attached to TM, the tangent bundle of M via the bundle morphism ρ. The vector bundle is so well attached that ρ is usually called the anchor map. The anchor map ρ and the Lie bracket have to be compatible: Together ρ and $[\cdot, \cdot$ satisfy the Leibniz rule. For our readers who are not familiar with this Lie algebroid world, we recall that each Lie algebra L is a Lie algebroid. The bracket is the bracket coming from the Lie algebra structure and ρ is the null map that sends $L \to \{0\} = TM$ (here M is a point).

For a Lie algebroid $(A \to M, [\cdot, \cdot], \rho)$, a naïve idea of ideal could be a subbundle $I \subset A$ such that $[\Gamma(I), \Gamma(A)] \subset \Gamma(I)$. If we look at the image of I by the anchor made, we get something fade, irrelevant, tasteless, without personality; in a word something *trivial*! (for those who like formulas, we obtain: $\rho(I) = 0$ or $I = A$).

This is where our scientist's work begins. The purpose is to find an alternative definition of ideals such that the image by the anchor map is not trivial. As in any history (or research work), our main character (or scientist) is not alone: in [2], the authors introduce the notion of *infinitesimal ideal system*. The idea behind infinitesimal ideal systems is the following. We go from a Lie group to a Lie algebra using the infinitesimal calculus and we can transpose this process from a Lie groupoid to a Lie algebroid. Starting from a Lie group G, we use the product of G to define a bracket m on its Lie algebra $\mathfrak{g}$. Then a candidate for an ideal subbundle is $K \subset \mathfrak{g}$ with $m(K, \mathfrak{g}) \subset K$. They do the the same constructions and see what happens for a Lie groupoid and its Lie algebroid. This leads to the definition of an *infinitesimal ideal system*.

An infinitesimal ideal system is a triple (F_M, J, ∇) where $F \subset TM$ is an involutive subbundle and $J \subset A$ is a subalgebroid such that $\rho(J) \subset F_M$ and $\nabla \colon \Gamma(F_M) \times \Gamma(A/J) \to \Gamma(A/J)$ is a flat connection on the quotient algebroid A/J. A section $a \in \Gamma(A)$ is said to be ∇-parallel if, for all $X \in \Gamma(F_M)$, we have $\nabla_X \tilde{a} = 0$, where $\tilde{a}$ is the class of a in $\Gamma(A/J)$. To have an infinitesimal ideal system, we require the following properties on ∇-parallel sections:

(1) For every ∇-parallel section a, we have $[a, j] \in \Gamma(J)$ for all $j \in \Gamma(J)$.
(2) The set of ∇-parallel sections is stable by the Lie bracket.

With this new definition of ideals, our scientist can travel around the world of the Lie algebroids to find them. The definition is very abstract and not easy

to check. That is why the main work of our scientist is to develop a tool to recognize infinitesimal ideal systems. Mathematically speaking, our scientist seek a necessary condition for infinitesimal ideal systems.

We will give a few details on this necessary condition. For explicit details on the following construction, we refer to [1]. Giving an infinitesimal ideal system (F_M, J, ∇), we can extend ∇ to obtain a new connection $\tilde{\nabla} \colon \Gamma(TM) \times \Gamma(A) \to \Gamma(A)$. Let $\mathcal{D}$ be the so called adjunct representation of $\tilde{\nabla}$, which is a flat connection of A with values in $A \oplus TM$. Using properties of infinitesimal ideal systems, we can show that $\mathcal{D}$ splits in two other flat connections. The first one, $\mathcal{D}_1$, is defined by restriction and goes from A to $J \oplus F_M$. The second one, $\mathcal{D}_2$, goes from A to $A/J \oplus TM/F_M$.

Each flat connection π has a Chern–Simons form $\mathrm{cs}(\pi, \pi^g)$. The goal is to see study relations between $\mathrm{cs}(\mathcal{D}, \mathcal{D}^g)$ and $\mathrm{cs}(\mathcal{D}_1, \mathcal{D}_1^g)$ and $\mathrm{cs}(\mathcal{D}_2, \mathcal{D}_2^g)$. This should lead to necessary conditions for infinitesimal ideal systems.

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Finding regularity for bound states in the N-body problem

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Communicated by *Rosa Marchesini*

There are structures which are recurrent in nature, laws describing interactions between infinitesimal small particles which also apply to the largest bodies we know to exist. Then we have common models, and of course common problems. The N-body problem is one of these. In most of its mathematical descriptions one mass is supposed to be the centre, the $(0, 0, 0)$ coordinate in the space $\mathbb{R}^3$. This can be for instance the kernel of an atom or the Sun in our Solar System. We should consider also other secondary bodies. In our examples, these are electrons and planets, respectively, which also have a position; let us say these are $x_i \in \mathbb{R}^3$, for each $i = 1, \ldots, N$. Since the central mass is fixed, we can describe the space configuration of the system by $x = (x_1, \ldots, x_N) \in \mathbb{R}^{3N}$. Physicists know that there are a lot of interactions which must be carefully summed up to describe the final behaviour of the system. The complexity and the quantity of problems involved imposes a choice. This is of course restrictive, but specializing is the key: the more we know about the single interactions, the more we know about the whole system.

We focus on the total energy of the system, which is relevant because it is time-invariant. It is usually described in quantum mechanics by the so called Schrödinger Hamiltonian, which can be expressed as the sum of operators corresponding to the kinetic energy and the potential energy of the particles involved:

(7.1) $$\mathcal{H} := -\Delta + V.$$

In the model we consider, the potential has the form

$$V(x) = \sum_{1 \leq i \leq N} \frac{b_i}{|x_i|} + \sum_{1 \leq i < j \leq N} \frac{c_{ij}}{|x_i - x_j|}$$

where $x = (x_1, \ldots, x_N) \in \mathbb{R}^{3N}$ as previously, $x_i \in \mathbb{R}^3$, and b_i and c_{ij} are suitable constants. $\mathcal{H}$ acts on $L^2(\mathbb{R}^{3N})$ with V as operator of multiplication.

Our main goal is to find results on the regularity of the eigenfunctions of $\mathcal{H}$, that is, of $u \in L^2(\mathbb{R}^{3N})$ such that

$$\mathcal{H}u := -\Delta u + Vu = \lambda u$$

for some $\lambda \in \mathbb{R}$. These special functions are also called bound states, due to their physical meaning: a particle in such a state cannot leave the system without additional energy.

What is the path we want to follow? The starting point is to define a manifold with good structures at infinity (a so called Lie manifold), starting from the manifold $X := \mathbb{R}^n$, with $n := 3N$, of all possible configurations of the system. We begin by applying the spherical compactification to X, which means adding a point at infinity for each half-line in $\mathbb{R}^n$ starting at the origin. In other words, we add a sphere $S_X := S^{n-1}$ of points at infinity. Let

$$Y_k := \{x \in \mathbb{R}^{3N} : x_k = 0 \in \mathbb{R}^3\} \subset \mathbb{R}^{3N}, \qquad Z_{ij} := \{x \in \mathbb{R}^{3N)} : x_i = x_j \in \mathbb{R}^3\}$$

The potential V is singular on the subspace

$$S := \bigcup_k Y_k \cup \bigcup_{i<j} Z_{ij}.$$

As usual, it is easier to work without singularities. We can avoid them by blowing X up along the submanifolds Y_k and Z_{ij} for all i, j, k with $i < j$ (see [1]). Roughly speaking, a new manifold $[X : S]$ is obtained by removing all the Y_k and Z_{ij} from X and, for each of these, gluing back the unit sphere bundle of their normal bundle in X. Actually, this is not sufficient for our purpose. We blow X up also along additional submanifolds of X, defined starting from the Y_k and the Z_{ij}. The resulting manifold $[X : S']$ turns out to be what we are looking for: the natural action of X by translations on X induces an action of X on $[X : S']$, which endows the blown-up manifold with a so-called Lie structure at infinity. Under this structure, we can define $\mathrm{Diff}([X : S'])$, an algebra of differential operators on $[X : S']$, where we can find an operator which is similar to (7.1) and has nice properties.

The second step is to extend and adapt a result in [1] to our case, which allows us to pass from operators in Lie manifolds to weighted Sobolev spaces, defined as

$$\mathcal{K}_a^m(\mathbb{R}^l) := \{u : \mathbb{R}^l \to \mathbb{C} : r_S^{|\alpha|-a} \partial^\alpha u \in L^2(\mathbb{R}^l), \ |\alpha| \le m\},$$

where $a \in \mathbb{R}$, $m \in \mathbb{N}$, and the weight $r_S(x)$ is a smoothed distance from x to S. We aim to prove the following claim under some condition on $a \in \mathbb{R}$ and $m \in \mathbb{N}$ yet to be determined:

$$u \in \mathcal{K}_a^m(\mathbb{R}^{3(N-1)})$$

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On the Numbers of Solutions of Systems of Homogeneous Forms

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Communicated by *Thorsten Hertl*

Since ancient times mathematicians have been interested in Pythagorean triples - even for practical purposes. Recall that a Pythagorean triple is a triple of integers (a, b, c) that satisfy $a^2 + b^2 = c^2$, or equivalently, a solution of the homogeneous quadratic form

$$a^2 + b^2 - c^2 = 0.$$

An easy example is $(2, 3, 5)$. But are there more? And how do we answer this question? If we put the trivial example $(0, 0, 0)$ aside we can divide the equation by c^2 and obtain the equation $p^2 + q^2 = 1$ over the rational numbers. In other words, we are looking for all rational points (p, q) on the unit circle. There are infinitely many points, which can be seen by the following argument: Let $(0, t)$ denotes the intersection of the ordinate with the line joining $(-1, 0)$ and (p, q). Then $t = q/(p+1)$ is a rational number and the hypothetical point (p, q) can be completely described by t via

$$(p, q) = \left(\frac{1 - t^2}{1 + t^2}, \frac{2t}{1 + t^2} \right).$$

Interpreting this formula the other way around, each rational t provides a rational point on the unit sphere and therefore a Pythagorean triple. Thus, there are infinitely many of them.

Nowadays, mathematicians do not restrict themselves to Pythagorean triples but want to understand the solution of (a system of) homogeneous forms. The general set up is the following: We are given R homogeneous polynomials $\mathbf{F} = (F_1, \ldots, F_R)$ in n variables $x = (x_1, \ldots, x_n)$ with coefficients in $\mathbb{Z}$. Since the theory becomes more accessible if all forms have the same degree d, we assume this, too. Unlike for Pythagorean triples, the problem is too complicated to expect a general recipe to describe all solutions. The next best thing is to estimate how many solutions there are under a given bound B. Formally speaking, we try to estimate

$$N(B) := \sharp\{x \in [-B, B]^n \cap \mathbb{Z}^n : \mathbf{F}(x) = 0\}$$

as well as possible. Heuristically, we expect $N(B)$ to grow as fast as B^{n-dR}, that is, $N(B)/B^{n-dR}$ is bounded from above and below by positive constants. Indeed, if we consider $\mathbf{F}$ as a polynomial in $n + 1$ variables (which is independent of the added variable) then we have produced $2B$-times many solutions more. To understand the dependence in R, it is best to work with forms of degree 1. Adding another form is like restricting the solutions to a hyperplane, which reduces the dimension by one. Lastly, it is harder to produce small numbers with high degree polynomials, so we expect B^{-d} many solutions.

A common theme to attack this problem is to look for solutions in the fields $\mathbb{Q}_p$ of p-adic numbers and in $\mathbb{R}$, and hope that such solutions lifts to solutions over the integers. If this works for all possible solutions, then we say that the system $\mathbf{F}$ satisfies the *local-to-global principle* or *Hasse principle*. Unfortunately, not every system $\mathbf{F}$ satisfies it. However, Birch [3] could show that good homogeneous systems support the heuristic and satisfy the Hasse principle:

Theorem 8.1. *Let $V = V(\mathbf{F})$ be the zero set of $\mathbf{F}$ and let V^* be set of all points, where the differential of $\mathbf{F}$ has no full rank. Assume that V is $n - R$ dimensional and is a complete intersection, that is, the intersection of exactly R hyperplanes. If*

$$(8.2) \qquad n - \dim V^* > 2^{d-1}(d-1)R(R+1),$$

then

$$(8.3) \qquad N(B) = \mathfrak{I}\mathfrak{S}B^{n-dR}(1 + o(1)).$$

If, in addition, V has a non-singular point over $\mathbb{R}$, then $\mathfrak{I} > 0$; if V has a non-singular point over $\mathbb{Q}_p$ for all primes p, then $\mathfrak{S} > 0$.

The proof relies on the *circle method*, a technique widely used in analytic number theory. It is based on the observation that

$$\int_0^1 \exp(2\pi i k t)\mathrm{d}t = \begin{cases} 1, & k = 0, \\ 0, & k \neq 0. \end{cases}$$

This observation allows us to express $N(B)$ as an integral. If we write $e(x)$ for $\exp(2\pi i x)$ and set

$$S(\alpha) := \sum_{x \in [-B,B]^n \cap \mathbb{Z}^n} e(\langle \alpha, \mathbf{F} \rangle),$$

then Fubini's theorem implies

$$
\begin{aligned}
\int_{[0,1]^R} S(\alpha) \, d\alpha &= \sum_{x \in [-B,B]^n \cap \mathbb{Z}^n} \int_{[0,1]^R} e(\langle \alpha, \mathbf{F} \rangle) \, d\alpha \\
&= \sum_{x \in [-B,B]^n \cap \mathbb{Z}^n} \int_0^1 \cdots \int_0^1 \prod_{j=1}^n e(\alpha_j F_j(x)) \, d\alpha_1 \ldots d\alpha_n \\
&= N(B).
\end{aligned}
$$

An involved analysis of the integral shows that the main contributions to the integral come from those $\alpha \in [0,1]^R$ that are close to rational points. All other points interfere with each other and roughly sum up to 0. More quantitatively, the *major arcs* of control-value Δ - to be determined later - is defined by

$$
\mathfrak{M}(\Delta) := \bigcup_{1 \leq q \leq B^\Delta} \bigcup_{\substack{a \bmod q \\ (a,q)=1}} \{\alpha \in [0,1]^R : 2||q\alpha - a||_\pi < B^{-d+\Delta}\},
$$

where $||x||_\pi = \min_{k \in \mathbb{Z}^n} |x - k|$. The complement is called the *minor arcs* and is denoted by $\mathfrak{m}(\Delta)$. One can show

$$
\int_{\mathfrak{M}(\Delta)} S(\alpha) \, d\alpha = \mathfrak{IS} B^{n-dR} + \mathcal{O}(B^{n-dR}),
$$

$$
\int_{\mathfrak{m}(\Delta)} S(\alpha) \, d\alpha = o(B^{n-dR-\delta}).
$$

This yields the claimed estimate for $N(B)$.

Recently, Myerson achieved a major breakthrough in [1]: He could replace Birch's quadratic lower bound by a linear one. The precise statement is as follows:

Theorem 8.4. *Let $\mathbf{F}$ be a generic system of length R of polynomials of degree d in n variables. Assume $n > (d2^d + 1)R$. Then*

$$
N(B) = \mathfrak{IS} B^{n-dR}(1 + o(1)).
$$

In addition, if V has a non-singular point over $\mathbb{R}$, then $\mathfrak{I} > 0$; if V has a non-singular point over $\mathbb{Q}_p$ for all primes p, then $\mathfrak{S} > 0$.

Around the same time, Schindler tackled the problem from a different angle. A homogeneous polynomial $F(x,y)$ with $x = (x_1, \ldots, x_{n_1})$ and $y = (y_1, \ldots, y_{n_2})$ is called *bihomogeneous* if it satisfies $F(\lambda x, \mu y) = \lambda^{d_1} \mu^{d_2} F(x,y)$. The bihomogenity allows us to ask the more refined question how

$$
N(B_1, B_2) := \sharp\{(x,y) \in ([-B_1, B_1] \times [-B_2, B_2]) \cap \mathbb{Z}^{n_1+n_2} : F(x,y) = 0\}
$$

asymptotically grows. One expects a Birch-style result for this set-up and this is indeed true. Schindler [2] proved such a theorem which reads (simplified) as follows:

Theorem 8.5. *Let V_1^* be the singular locus for $\mathbf{F}$ in x direction and V_2^* let be the singular locus in y direction. Assume they satisfy*

$$
n_1 + n_2 - \dim V_i^* > Q(R),
$$

where $Q(R)$ is a specifically chosen quadratic polynomial. Then

$$N(B_1, B_2) = \mathfrak{S}\mathfrak{J}B^{n_1 - d_1 R}B^{n_2 - d_2 R}(1 + o(1))$$

and a version of the Hasse principle holds.

Now the question arises whether one can replace the quadratic lower bound by a linear one. This is precisely one of Leonhard's PhD projects. Although he just started in August this year he already obtained a result for bilinear forms, the $(d_1, d_2) = (1, 1)$ case. The next step - bidegree = $(2, 1)$ - requires more combinatorics, but we are sure that this case as well as the general case will be proven.

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New techniques for the study of singular foliations

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Communicated by *Fabrizio Zanello*

Regular and Singular Foliations. Leonid's work is concerned with the study of the topological and algebraic properties of singular foliations. Before describing some of the advanced mathematical tools involved in his research, it is instructive to retrace the path to the correct definition of what a foliation is. The most intuitive definition of a *regular* foliation is probably the following:

Definition 9.1. Given a manifold M of dimension n, a regular foliation is a decomposition of M into submanifolds $M = \bigsqcup_{\alpha \in \Lambda} L_\alpha$, where Λ is some index set, such that in local coordinates the submanifolds are diffeomorphic to $\mathbb{R}^k \times \{\alpha\} \subset \mathbb{R}^n$.

But soon there appeared some particular situations allowed by this definition which are not well suited for the purposes of Leonid's subsequent work. An example is the following decomposition of $\mathbb{R}^2$:

$$\mathbb{R}^2 = \{x = 0\} \bigsqcup \left(\bigsqcup_{\alpha \in \mathbb{R}} \{(x, \alpha) : x \neq 0\} \right).$$

The problem in this case is that the leaf $\{x = 0\}$ is *transverse* to the other leaves $\{(x, \alpha) : x \neq 0\}$. In order to avoid such foliations, a stricter definition has been introduced:

Definition 9.2. A regular foliation is a subbundle $V \subseteq TM$ which is also *involutive*; in other words, if we denote by $\Gamma(V)$ the space of sections of V, then $[\Gamma(V), \Gamma(V)] \subset \Gamma(V)$.

Since it will be relevant for the approach to singular foliations, we also state another equivalent definition with a more algebraic taste:

Definition 9.3. A regular foliation $\mathfrak{F}$ is a subset of the space of vector fields $\mathfrak{F} \subset \mathfrak{X}(M)$ such that

- $[\mathfrak{F}, \mathfrak{F}] \subset \mathfrak{F}$;
- $\mathfrak{F}$ has constant rank, that is, the vector spaces $\mathfrak{F}_x = \mathfrak{F} \bigcap T_x M$, $x \in M$, have constant dimension;
- $\mathfrak{F}$ is a $C^\infty(M)$-submodule of $\mathfrak{X}(M)$.

Now we are ready to define singular foliations, which generalize regular foliations. First, we may weaken our first definition of a regular foliation:

Definition 9.4. A singular foliation is a decomposition $M = \bigsqcup_\alpha L_\alpha$ where the leaves L_α may have different dimensions.

This definition has to be discarded immediately because it also allows the unsuitable example mentioned above. A condition that should be added is *extensibility*: we ask that for every tangent vector $v \in T_x L_\alpha$ there is a local vector field X (that is, a vector field which is defined in a neighbourhood of x) such that $X|_x = v$.

If Leonid had to work in a purely geometrical setting this would be enough. This further requirement is, however, still not enough for the algebraic structures that he is interested in. The solution is to modify Definition 9.3 in the following way. It is understood that involutivity is something that we really cannot give up. So we drop the assumption about the constancy of the rank and keep only the hypothesis that $\mathfrak{F}$ is a $C^\infty(M)$-submodule of $\mathfrak{X}(M)$; then we add the algebraic requirement that $\mathfrak{F}$ has to be *finitely generated* as a $C^\infty(M)$-module.

To summarize the geometric and algebraic approaches, we have the following table:

	regular foliations	singular foliations
geometric	$V \subseteq TM$ involutive subbundle	$M = \bigsqcup_\alpha L_\alpha$ with embedded submanifolds, possibly with different dimensions, every $v \in T_x L_\alpha$ has a local extension.
algebraic	$\mathfrak{F} \subset \mathfrak{X}(M)$ involutive $C^\infty(M)$-submodule with constant rank	$\mathfrak{F} \subset \mathfrak{X}(M)$ involutive, finitely generated $C^\infty(M)$-submodule.

The geometric and the algebraic definitions for regular foliations are equivalent by the *Frobenius Theorem*. In contrast, for singular foliations, the algebraic definition is only a special case of the less restrictive geometric one.

The Lie ∞-Algebroid Structure. Starting from the fact that an admissible foliation has to be finitely generated, it is possible to define a sequence of vector bundles.

Suppose that a foliation $\mathfrak{F}$ admits N vector fields as generators. Let us introduce the vector bundle $E_{-1} \coloneqq \mathbb{R}^N \times M \to M$. Then, fixing a system of generators $\{v_1, \ldots, v_N\}$ of the foliation $\mathfrak{F}$, we get a map $r \colon \Gamma(E_{-1}) \longrightarrow \mathfrak{F}$. In general, this map need not be injective. Then $\mathrm{Ker}(r) \neq \{0\}$. Assume that this module is, in turn, finitely generated by, say, k elements. Then, as above, we get a vector bundle $E_{-2} \coloneqq \mathbb{R}^k \times M \to M$ and a map $\Gamma(E_{-2}) \to \Gamma(E_{-1})$ whose image is $\mathrm{Ker}(r)$. This lifts to a vector bundle map $E_{-2} \to E_{-1}$. Repeating this construction, we get a long exact sequence

$$\cdots \to \Gamma(E_{-n}) \to \cdots \to \Gamma(E_{-2}) \to \Gamma(E_{-1}) \to \mathfrak{F}.$$

This is a *projective resolution* of $\mathfrak{F}$. In general, $\mathrm{Ker}(r)$ and the kernels of the other maps in the construction just mentioned need not to be finitely generated, even if $\mathfrak{F}$ is. Nevertheless, this is the case in all the situations considered in Leonid's research work, and so the procedure above works.

On each space of sections of each vector bundle there is a bracket (due to the involutivity requirement for the admissible foliations), which implements on the sequence, considered as a single object, the structure of *Lie ∞-algebroid*. This is a weaker and more general version of the structure of Lie algebra. Lie algebroids are Lie ∞-algebroids where the sequence above is reduced to a single non-trivial map $\cdots \to 0 \to \Gamma(E_{-1}) \to \mathfrak{F}$. The Lie ∞-algebroid associated to a foliation is called the *universal* Lie ∞-algebroid of the foliation.

Results and Outlook. One of the main results states that this construction is natural, in the sense that *properly defined invariants of the universal Lie ∞-algebroid of a foliation are invariants of the foliation itself.* Moreover, the formulation of this theory in the framework of Lie ∞-algebroids allowed to recover fundamental results about Lie algebroids and singular foliations.

One of the strong points of the theory is the possibility to use the invariants defined through the universal Lie ∞-algebroid to distinguish different foliations of the same space. This technique leads to many questions, which Leonid and his collaborators will try to answer, such as the following.

- When can the universal Lie ∞-algebroid of a foliation be generated by a Lie algebroid? In other words, when is the projective resolution of $\mathfrak{F}$ reduced to a single map $\mathfrak{B} \to \mathfrak{F}$?
- How deep are the topological and the algebraic aspects and how far can we separate and control their contributions to the invariants of a foliation?

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Field theories associated to the Sine-Gordon equation

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Communicated by *Leonid Ryvkin*

Introduction: The Sine-Gordon equation. Fabrizio's research centers on the famous Sine-Gordon equation. It is a partial differential equation for real-valued functions in two variables $\varphi = \varphi(t, x)$, spelling out as follows:

$$(10.1) \qquad \frac{\partial}{\partial t}\frac{\partial}{\partial t}\varphi - \frac{\partial}{\partial x}\frac{\partial}{\partial x}\varphi + \sin(\varphi) = 0.$$

This equation appears naturally in the theory of negatively curved Riemannian surfaces and has several notable properties:

- There is a systematic procedure to generate a series of new solutions from an old one. In particular, there are many solutions.
- There are infinitely many *conserved quantities*, that is, "invariants" of solutions.
- The equation admits soliton solutions. If we interpret t as time and x as a spacial direction, then we can find solutions with "one isolated wave moving in space as time passes".

These three properties (many solutions, many invariants, interesting solutions) make it an interesting PDE to study.

Phasing PDEs physically: Field Theory. One possible way to interpret the Sine-Gordon equation is as a *Lagrangian classical field theory*. The basic setup of such a field theory is given by

- a spacetime M;
- a space of fields $\mathcal{C}$;
- a Lagrangian L.

In our case, the spacetime is $M = \mathbb{R}^2$, interpreted as a 2-dimensional version of spacetime with one space direction x and one time direction t. Typically, a field $\varphi \in \mathcal{C}$ associates a physical quantity $\varphi(t, x)$ to any point (t, x) in spacetime. In general, a field may be valued in any space (of vectors, tensors, group elements) or even a fibration of spaces over M. In our case, it simply takes real values. For this reason the Sine-Gordon equation (or model) is also called a *scalar* field theory. For a scalar classical field theory describing a second-order PDE (like the

Sine-Gordon-Equation), L depends on the spacetime coordinates (t, x), the field value φ and its first derivatives (φ_t, φ_x). In our case

$$L(t, x, \varphi, \varphi_t, \varphi_x) := \frac{1}{2}(\varphi_t^2 - \varphi_x^2) + (1 - \cos(\varphi)).$$

As the Lagrangian only depends on zero and first order derivative coordinates, the Sine-Gordon model is a *first-order* classical field theory.

The idea of the Lagrangian, is to translate a PDE into a minimization problem. For instance, up to convergence problems, the solutions of the Sine-Gordon equation (10.1) are equivalent to the extremities (for instance, local minima) of the functional

$$\varphi \mapsto \int L\Big(t, x, \varphi(t, x), \frac{\partial \varphi}{\partial t}(t, x), \frac{\partial \varphi}{\partial x}(t, x)\Big) \, \mathrm{d}t \, \mathrm{d}x.$$

Algebraic Approach: Constructing Quantisations. The Lagrangian L defines a Lie bracket $\{\cdot, \cdot\}$ on (an appropriate subspace of) the smooth functionals $\mathcal{F} \subset C^\infty(\mathcal{C}, \mathbb{C})$ on the field space $\mathcal{C}$. A functional $F \in \mathcal{F}$ maps a field φ to a number $F(\varphi)$. Together with the pointwise (commutative) multiplication of functionals, the Lie bracket $\{\cdot, \cdot\}$ turns $\mathcal{F}$ into a *Poisson algebra*.

Roughly speaking, the idea of quantization is that the classical field theory or PDE (10.1), described through the Poisson algebra $\mathcal{F}$, is the "large-scale behaviour" of our system, that is, the result of some parameter approaching zero. From this perspective, $\mathcal{F}$ is just the shadow or *classical limit* of a richer structure on $\mathcal{F}[[\hbar]]$, the space of formal power series with coefficients in $\mathcal{F}$ in the formal variable $\hbar$, to be interpreted as the Planck constant (a very small number). From this perspective, the quest for a *quantization* of the Sine-Gordon equation translates to finding a product structure $*_\hbar \colon \mathcal{F}[[\hbar]] \times \mathcal{F}[[\hbar]] \to \mathcal{F}[[\hbar]]$, which satisfies the following:

- the zero-order part of $*_\hbar$ recovers the point-wise multiplication on $\mathcal{F}$, that is,
$$\lim_{\hbar \to 0} F *_\hbar G = F \cdot G;$$

- the first-order part of the commutator of $*_\hbar$ recovers the Lie bracket of $\mathcal{F}$, that is,
$$\lim_{\hbar \to 0} \frac{F *_\hbar G - G *_\hbar F}{\hbar} = \sqrt{-1}\{F, G\}.$$

Introducing interactions. The Sine-Gordon equation may be done solved by a *perturbative approach*, as it is closely related to the similar but simpler Klein-Gordon equation:

$$\frac{\partial}{\partial t}\frac{\partial}{\partial t}\varphi - \frac{\partial}{\partial x}\frac{\partial}{\partial x}\varphi + \varphi = 0.$$

We interpolate between these equations and their Lagrangian by a parameter κ:

$$L_\kappa(t, x, \varphi, \varphi_t, \varphi_x) := \frac{1}{2}(\varphi_t^2 - \varphi_x^2) + (1 - \kappa)\frac{1}{2}\varphi^2 - \kappa(\cos(\varphi) + 1).$$

For $\kappa = 0$, we get the Klein-Gordon equation, which is called the *free theory*. For $\kappa = 1$, we get the Sine-Gordon equation, the so-called *theory with interaction*. This

interpolation yields a Poisson algebra $\mathcal{F}[[\kappa]]$, which relates the functional algebras of the Klein–Gordon and the Sine-Gordon equation.

Perturbative algebraic quantum field theory. Fabrizio's research aims at understanding how the quantization and the perturbation procedures can be carried out consecutively, yielding a product on the space $\mathcal{F}[[\kappa, \hbar]]$. While for quantization followed by perturbation, there are some results, the converse direction of "how to quantize the perturbed theory" is completely open.

$$\begin{array}{ccc} \text{unperturbed classical theory } \mathcal{F} & \longrightarrow & \text{perturbed classical theory } \mathcal{F}[[\kappa]] \\ \downarrow & & \downarrow ? \\ \text{unperturbed quantum theory } \mathcal{F}[[\hbar]] & \longrightarrow & \text{perturbed quantum theory } \mathcal{F}[[\hbar, \kappa]] \end{array}$$

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A very complicated L^2-invariant

Thorsten Hertl

Communicated by *Leonhard Hochfilzer*

Let (M, g) be a smooth Riemannian manifold of dimension d. There are several ways to define the curvature of a Riemannian manifold. Three important definitions of the curvature are the Riemann curvature tensor, the Ricci curvature tensor and the scalar curvature. When $d = 2$, these notions coincide in the sense that knowing one of the three allows to compute the other two. While this is no longer the case for $d > 2$, one may still deduce the Ricci tensor from the Riemann tensor, and the scalar curvature from the Ricci tensor.

The main object of interest for us here is the scalar curvature. As a first definition, the scalar curvature is usually defined to be the trace of the Ricci curvature tensor. However, there is also a direct definition that does not require us to compute the Ricci tensor first. Recall that a Riemannian metric g induces a metric d_g on our manifold M. We define the open ball of radius r in M to be

$$B_r(p) := \{q \in M : d_g(p, q) < r\}.$$

Definition 11.1. Let (M, g) be a smooth Riemannian manifold of dimension d. We define the scalar curvature of (M, g) at a point $p \in M$ to be the real number $\mathrm{scal}_g(p)$ such that the following holds

$$\frac{\mathrm{vol}(B_\varepsilon(p) \subseteq M)}{\mathrm{vol}(B_\varepsilon(0) \subseteq \mathbb{R}^n)} = 1 - \frac{\mathrm{scal}_g(p)}{6(d+2)}\varepsilon^2 + O(\varepsilon^4).$$

Thus the scalar curvature measures the asymptotic growth of geodesic balls compared to the growth of balls in Euclidean space. In the case $d = 2$ the scalar curvature is 2κ, where κ is the Gauss curvature.

Example 11.2. Consider the unit ball $M = S_1^2 = \{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 + z^2 = 1\}$ of dimension 2 with its usual metric. Let $a \in M$. It is not hard to show that $\mathrm{vol}(B_\varepsilon(a) \subseteq M) = 2\pi(1 - \cos\varepsilon)$. The Taylor expansion of $\cos\varepsilon$ implies $\mathrm{vol}(B_\varepsilon(a) \subseteq M) = \pi\varepsilon^2 - \frac{\pi}{12}\varepsilon^4 + O(\varepsilon^6)$. Recalling that the area of a disc with radius ε is $\pi\varepsilon^2$ we see that $\mathrm{scal}_g(a)$ is positive in this case (as we hope it would be). In fact, we find $\mathrm{scal}_g = 2$.

We say that a metric g has *positive scalar curvature* if $\mathrm{scal}_g(a) > 0$ for all $a \in M$. The following theorem by Kazdan and Warner motivates the study of positive scalar metrics.

Theorem 11.3 (Kazdan, Warner, 1975)**.** *Let M be a closed smooth manifold of degree $d \geq 3$. Assume that there is a positive scalar curvature metric on M. Then for all $f \in C^\infty(M)$ there is a metric g_f such that $f = \mathrm{scal}_{g_f}$.*

If $d = 2$ then the Gauss–Bonnet Theorem can help us study whether a surface M admits a positive scalar metric. Recall that $\mathrm{scal}_g = 2\kappa_g$, where κ_g is the Gauss curvature. The Gauss–Bonnet Theorem may thus be stated as the equation

$$\int_M \mathrm{scal}_g \, \mathrm{d}(\mathrm{vol}_g) = 4\pi\chi(M),$$

where $\chi(M)$ denotes the Euler characteristic of M. Any surface homeomorphic to a torus has Euler characteristic $\chi(M) = 0$. Therefore, such a manifold has no positive scalar curvature metric.

For $d \geq 3$ things become more difficult. For so-called spin-manifolds there is the *α-invariant* denoted by $\alpha(M)$. It is a topological invariant, but it can also be constructed using a metric g – similar to the Euler characteristic in the above setting. One can show that a positive scalar curvature metric can only exist if $\alpha(M) = 0$.

The next step is to ask how many positive scalar curvature metrics there are. To be more precise, one defines the set

$$R^+(M) \coloneqq \{\text{positive scalar curvature metrics on } M\}.$$

It comes with a natural topology. One topological interpretation of how many positive scalar curvature metrics there are is to examine the richness of the homotopy groups $\pi_k(R^+(M))$.

Recall that two metrics g_0 and g_1 are *isotopic* if there is a family of metrics $g_t \colon [0,1] \to R^+(M)$ that varies smoothly in t. Write $[g]$ for the class of metrics that are isotopic to g. It turns out that

$$\pi_0(R^+(M)) = \{[g] \colon g \text{ is a positive scalar curvature metric on } M\}.$$

In general, finding nontrivial elements in the homotopy groups $\pi_k(R^+(M))$ is hard. Similar to the α-invariant above, there is a *relative α-invariant*, denoted α_{rel}, which depends on the dimension of the manifold and the homotopy group that we use for it. For example, if we consider $k = 0$ then $\alpha_{\mathrm{rel}}(g_0, g_1) = 0$ if g_0 and g_1 are isotopic.

Another approach to studying positive scalar curvature metrics further is to study them up to concordance. Two metrics g_0 and g_1 are called concordant if there is a metric $\tilde{g} \in R^+(M \times [0,1])$ such that $\tilde{g} = g_j + \mathrm{d}t^2$ for $j = 0, 1$ near the boundary of $M \times [0,1]$. Note that isotopy implies concordance. The reverse implication is still an open question. One of Thorsten's recent achievements was to construct a set $\tilde{R}^+(M)$ such that

$$\pi_0(\tilde{R}^+(M)) = \{\text{concordance classes of positive scalar curvature metrics on } M\}.$$

Somewhat by accident, the fundamental group $\pi_1(\tilde{R}^+(M), g_0)$ is in bijection with self-concordance classes of g_0 of positive scalar curvature metrics on M.

The reason to study $\tilde{R}^+(M)$ is that α_{rel} is difficult to compute for explicit examples. In particular, it is not known whether α_{rel} might depend only on the concordance class as opposed to the isotopy class. Studying $\tilde{R}^+(M)$ is an attempt to tackle this problem by considering the corresponding construction for $\pi_k(\tilde{R}^+(M))$. Finally, another motivation is that we have an injection $R^+(M) \hookrightarrow \tilde{R}^+(M)$. It might be easier to detect non-trivial elements of $\pi_k(\tilde{R}^+(M))$, which could then lead to a tool for finding non-trivial elements of $\pi_k(\tilde{R}^+(M))$.

Introduction to the circle method

ROK HAVLAS

Communicated by *Zhicheng Han*
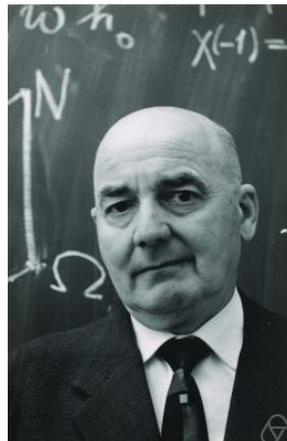
Consider the following equation:

$$x^2 + y^2 + z^2 + w^2 = N$$

Can we solve them with integral solutions $(x, y, z, w) \in \mathbb{Z}^4$ for any natural number $N \in \mathbb{N}$?

The first affirmative answer to this question is due to *Joseph-Louis Lagrange* back in 1770, known today as **Lagrange's 4-square theorem**

In the same year, *Edward Waring* published his famous monograph **Meditationes Algebraicae**, in which he asserted that every natural number is the sum of at most 4 squares, or 9 cubes, or 19 biqudrates (which means fourth powers, and similiarly *tesseratics* as five power on, etc.), and so on. He also formulated the the following **Waring's Problem**:

Conjecture 12.1 (Waring). *For every $k \in \mathbb{N}$, there is a respective $s \in \mathbb{N}$ such that every $N \in \mathbb{N}$ can be written as a sum of s-many k-th powers, i.e., the equation:*

$$x_1^k + x_2^k + \cdots + x_s^k = N$$

has integral solutions for some $(x_1, \cdots, x_s) \in \mathbb{Z}^s$.

After more than one century, This conjecture was finally answered by *David Hilbert* in 1909 [Hil09]. Waring's problem is nowadays also known as **Hilbert-Waring Theorem**.

In 1920, *G.H. Hardy* and *John Edensor Littlewood* gave a new proof of Warring's problem using a new method, today knwon as **Hardy-Littlewood circle method**. In a nutshell this proof gave an asymptotic formula to the number of ways to write large nautral number as a sum of s-many k-th powers.

The circle method is still a vital tool in number theory to this very day (after a full century!) To understand the breadth of problems it can address, let us shift our attention momentarily and consider another problem called **Hasse Principle**. Consider the following problems:

(1) Does the equation $x^2 + y^2 = -1$ admit any integral solutions?
(2) Does the equation $x^2 - 3y^2 = 2$ admit any integral solutions?

The answer is negative to both questions. For the first problem, one may easily check there is no real solution in the first place. For the second, one argue by contrapositive: Suppose x is a solution , then $x^2 \equiv 2 (\mod 3)$, but this is not possible, as for all integers, $x^2 \equiv 0$ or 1 modulo 3. Succinctly we may summarize these two 'no's as obstructions to the existence of integral solutions:

Condition I: There is a real solution to the equation;
Condition II: for each $n \in \mathbb{N}$ there is a solution to the equation $\mod n$.

The above $(\mod p)$-methodology can be applied to any prime p. If we consider ∞ as a prime as well, then Condition I can also be seen as $(\mod \infty)$-solution.

Similarly we may ask the converse question:

Does these conditions also give a suffcient condition for the existence of integral solutions?

The above equation is said to admit a $\mathbb{Q}_p$-solution if it can be solved (mod p). One should really think these $\mathbb{Q}_p$ as the 'local' part of information (for which it earns the name **local fields**, as opposed to their $\mathbb{Q}$-counterpart which are called **global fields**.) Moreover, we call those cases for which the conditions are suffcient that the **Hasse-Principle** holds. This in turn is also called **local-to-global principle**.

We give an example and a counterexample for Hasse principle. The Hasse principles holds for quadratic forms. More precisely, we have the following theorem:

Theorem 12.2 (Hasse-Minkowski)**.** *Let $F \in \mathbb{Z}[x_1, \cdots, x_s]$ a quadratic form Then $F = 0$ has non-trivial integral solutions if and only if it has real solutions, and $\mathbb{Q}_p$-solutions for all prime ps.*

Note the result cannot be extended to any forms of higher degree. A famous counterexample was due to *Ernst Selmer* in 1951:

Example 12.3 ([Sel51])**.** The equation $3x^3 + 4y^3 + 5z^3 = 0$ has real solutions and $\mathbb{Q}_p$-solutions but does not admit integral solutions.

The obstruction to the validity of Hasse principles is gauged by **Brauer-Manin Obstruction**, namely: Hasse principle does not hold true for varieties with non-trivial associated Brauer group. (For brevity we shall not discuss what the Brauer group is. One could think it as the equivalence class of central simple algebras.) Using Brauer-Manin obstruction **Peter Swinnerton-Dyer** in 1962 constructed a counterexample in degree 4 [SD62]:

$$7x_1^4 + 8x_2^4 - 9x_3^4 - 14x_4^4 = 0$$

The reader should note that Brauer-Manin obstructions cannot fully determines validity of Hasse principle. On one hand , **Alexei Skorobogatov** further proved in [Sko99] that Brauer-Manin obstruction does not measure the full degree of failure, that is, there are cases when Brauer-Manin obstruction vanishes, yet Hasse principles still fails to hold. On the other hand, **Jean-Louis Colliot-Thélène** proved in [PV04] that Brauer-Manin obstruction is VOID for any non-singular hypersurface in more than $\mathbb{P}^n$ for $n \geq 5$

Let use return to the beginning of this question. Having showed the failure of Hasse-principle for cubic and quartic forms in 3-variables and 4-variables respectively, one would ask another sensible question:

what would be the least number of variables for which the Hasse principles holds?

In cubic forms, the best lower bound is known to be 14, whereas in quartic forms *Oscar Marmon* and *Pankaj Vishe* [MV19] proved in 2019 that the best bound could be refined to 29. **All these refinements used circle methods**.

So, what is circle method?

To understand it, first consider a system of polynomials $(f_1, \ldots f_s)$ of n variables, that is, $f_i \in \mathbb{Z}[X_1, \cdots, X_n]$. Let $\mathcal{B} \subseteq \mathbb{R}^n$ be a box with sides parallel to

coordinate axes. We also denote $B\mathcal{B}$ be the dilation of the dilation of the box by the factor $B \geq 1$. Now we denote:

$$N(B) = \#\{X \in \mathbb{Z} \cap B\mathcal{B}\}$$

the number of integral solutions to the system of polynomials. Suppose further all f_i are homogeneous and cut out a nonsingular complete intersection zero set $X \subseteq \mathbb{P}^{n-1}$. To show the variety $f_1 = \cdots = f_s$ has an integral point, we suffices to show that $N(B) > 0$ for B sufficiently large, where we also choose $\mathcal{B}$ accordingly.

Define $e(z) := e^{2\pi i z}$ the suitably normalized function on $z \in \mathbb{R}$. For any $z \in \mathbb{R}$. now by orthogonality, we have:

$$N(B) = \sum_x \int_0^1 e(\alpha F(x)) d\alpha = \int_0^1 \sum_x e(\alpha F(x)) d\alpha$$

which we denote the integrand as $S(\alpha)$. Hence we can alternatively take:

$$S(\alpha) = \sum_{x \in \mathbb{Z}^n \cap B\mathcal{B}} e(\alpha_1 f_1(x) + \cdots + \alpha_n f_n(x))$$

that is, the number of solution becomes integration of some exponential sum. Now the circle method enters the scene by giving an apt asymptotic analysis of the the integral as $B \to \infty$. In so doing, we split the unit circle (as $e(z)$ sweeps through the unit interval) into *major arc*, which accounts for the major contribution, and *minor arc*, which can be considered as 'noise' occurred during applying Fourier analysis.

Now if X is a complete intersection of s-many hypersurfaces of deg $d_1 \cdots d_s$ such that X is non-singular and projective, then we may assume both major and minor arcs are well-behaved like aforementioned with $\prod_\nu \sigma_\nu$ absolutely convergent, from which we can conclude the Hasse Principle holds for X. Indeed, $X(\mathbb{R}) \neq 0$ implies $\mathcal{B}$ is a small box centered around a nonsingular point, which implies the real density $\sigma_\infty > 0$. In similar fashion, $X(\mathbb{Q}_p) \neq 0$ for all p also implies $\prod_p \sigma_p > 0$, that $N(B) > 0$ for B sufficiently large altogether implies $X(\mathbb{Q}) \neq 0$.

Another perspective of this method is rather probabilistic. As α runs through the unit cube, we expect the exponential $e(\alpha_1 f_1 + \cdots + \alpha_s f_s)$ to be rather randomly scattered around the unit circle as we perturb $x \in \mathbb{Z}^n$, this brought another version of **central limit theorem**: $S(\alpha)$ should be roughly of order $B^{n/2}$.

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Representing homology classes as manifolds

THORBEN KASTENHOLZ

Communicated by *David Kern*

This article is about Thorben Kastenholz' field of research, which is in the intersection of the algebraic topology of manifolds and differential topology. Up to now, he was working with Ursula Hamenstädt at Bonn University, where he handed in his PhD thesis in September 2020. In short, he is interested in the minimal genus of surface bundles and the corresponding maps. In the following there will be a short introduction to his latest problems.

Let M be a smooth, oriented and compact manifold of dimension $2k$ for $k \in \mathbb{N}$. A bilinear form $(\cdot, \cdot)$ on the de Rham cohomology $\mathrm{H}^k(M; \mathbb{R})$ is defined by

$$(\alpha, \beta) = \int_M \alpha \wedge \beta$$

for $\alpha, \beta \in H^k(M; \mathbb{R})$. If k is odd, then the bilinear form $(\cdot, \cdot)$ is antisymmetric. If k is even, it is symmetric and nondegenerate by Poincaré duality. Then the vector space $H^k(M; \mathbb{R})$ splits in the subspaces V_+ and V_- where $(\cdot, \cdot)$ is positive and negative definite, respectively. The dimensions of these subspaces yield the *signature* of M:

$$\sigma(M) = \dim(V_+) - \dim(V_-).$$

This is an invariant of M.

Two manifolds M and N are called *bordant* if there is another manifold W such that

$$\partial W = M \sqcup \overline{N},$$

where $\overline{N}$ denotes N with the opposite orientation. If two manifolds are bordant, then they have the same signature.

For a manifold M as before, another important invariant is the *Euler characteristic* defined by

$$\chi(M) = \sum_{i=1}^{k} (-1)^i \dim H^k(M; \mathbb{R}),$$

where k is the dimension of M. If $k = 4l$, the Euler characteristic and the signature are related by $\sigma(M) = 2\chi(M)$. Let $M \to E \to N$ be a fibre a bundle, that is, a bundle with base manifold N whose typical fibre is the manifold M. Then E is a manifold as well. The Euler characteristic of E may be computed as the product $\chi(E) = \chi(M)\chi(N)$.

There is, however, no such formula for the signature σ of a fibre bundle. For instance, $\sigma(E) \neq \sigma(M)\sigma(N)$ for the fibre bundle $\Sigma_g \longrightarrow E \longrightarrow \Sigma_h$ with $\sigma(E) \neq 0$, where g and h denote the genera of the base manifold and the typical fibre. This is surprising because for trivializable fibre bundles and for fibre bundles with simply-connected base manifold N, the signature turns out to be multiplicative. This leads to one of the central questions.

Question 13.1. Fix the genus g. What is the minimal genus of Σ_h, such that a fibre bundle $\Sigma_g \longrightarrow E \longrightarrow \Sigma_h$ with $\sigma(E) \neq 0$ exists?

Let X be a path-connected topological space and let Σ_g be a surface of genus g. The following is well-known: for each $\alpha \in H_2(X;\mathbb{Z})$ there is a continuous map $f \colon \Sigma_g \longrightarrow X$ such that $f_*[\Sigma_g] = \alpha$. This leads to the second central open question:

Question 13.2. What is the minimal genus g such that $\alpha = f_*[\Sigma_g]$ for a given $\alpha \in H_2(X;\mathbb{Z})$?

In the following, we assume $g \geq 5$. Let $B\mathrm{Diff}(\Sigma_g)$ denote the classifying space of $\mathrm{Diff}(\Sigma_g)$. The maps from a surface topological space X into $B\mathrm{Diff}(\Sigma_g)$ are in bijective correspondence with fibre bundles $\Sigma_g \longrightarrow E \longrightarrow X$, that is,

$$(13.3) \qquad\qquad [X, B\mathrm{Diff}(\Sigma_g)] \cong \{\Sigma_g \longrightarrow E \longrightarrow X\}.$$

On the one hand, the second cohomology of $B\mathrm{Diff}(\Sigma_g)$ is isomorphic to the integers $\mathbb{Z}$. On the other hand, it is isomorphic to $\mathrm{Hom}(H_2(B\mathrm{Diff}(\Sigma_g)), \mathbb{Z})$. Using the isomorphism (13.3), one can define a signature map from $H_2(B\mathrm{Diff}(\Sigma_g))$ to $\mathbb{Z}$, mapping f to $\sigma(E_f)$, where E_f is the fibre bundle corresponding to f by (13.3). The signature map, which we also denote by σ, generates all the maps in $\mathrm{Hom}(H_2(B\mathrm{Diff}(\Sigma_g)), \mathbb{Z})$. This leads to another interesting question:

Question 13.4. What is the minimal g for which there is a generator τ of $H_2(B\mathrm{Diff}(\Sigma_g))$?

In this homological framework, the isomorphism (13.3) leads to another interesting question:

Question 13.5. What is the minimal genus of a sphere bundle $\Sigma_g \longrightarrow E \longrightarrow \Sigma_h$, such that $\sigma(E) = 4\lambda$? **What is λ?**

Of course, one may wonder why we should restrict ourselves to surface bundles. Why hot study fibre bundles with base manifolds and typical fibres of higher dimensions? Some basic statements about the four questions above break down. For example, for a higher-dimensional topological space X and $\alpha \in H_2(X, \mathbb{Z})$, there need not be a continuous map $f \colon \Sigma_g \longrightarrow X$ such that $f_*[\Sigma_g] = \alpha$. Thus it becomes a lot more difficult to even state the right questions. This is also part of the work in progress.

Geometric quantization – What are the physicists doing?

DAVID KERN

Communicated by *Thorben Kastenholz*

David holds a Bachelors Degree in electrical engineering from the University of Ulm and a Masters Degree in Mathematical Physics from the University of Würzburg. Since 2019, he is a PhD student in Göttingen and supervised by Madeleine Jotz Lean, Ralf Meyer and Chenchang Zhu. His thesis subject is closely related to his background from physics.

In classical mechanics, the system in question is described by a symplectic manifold or more general by a Poisson manifold. In this case, points of this manifold correspond to states of the physical system and smooth functions from the manifold to $\mathbb{R}$ correspond to observables like place or energy. The smooth Hamiltonian vector field (the Hamiltonian in the Poisson manifold sense of the Hamiltonian energy function) yields a smooth flow on the manifold, which represents the time evolution of the physical system. Infinitesimally, the time evolution of observables is described by the Poisson bracket of a function ("observable") with the Hamiltonian energy function.

Physicists have a way, called quantization, to associate to a classical mechanical system a quantum mechanical system. The states of such a system are described by a Hilbert space. Self-adjoint operators on this Hilbert space correspond to observables and the infinitesimal time evolution of observables is described by the Lie bracket of a self-adjoint operator/observable and the Hamiltonian energy functional.

The harmonic oscillator is an illustrative example. Classically, it is described by an ellipse, whereas quantum mechanically, the Hilbert space is given by $L^2(\mathbb{R})$. In both cases, the periodic behaviour of the system is dictated by the description of the Hamiltonian energy function and the resulting time evolution.

While physicists are capable of replacing a classical mechanical system by its quantum mechanical analogue, mathematicians struggle to understand what physicists are actually doing. Or in more technical terms: How does one assign a Hilbert space to a Poisson manifolds in such a way that functions correspond to self-adjoint operators and Poisson brackets to Lie brackets? Most importantly, the construction should represent the physical quantization process.

David is studying one of the candidates for a mathematical description of quantization, the so called *geometric quantization*, which consists of three steps. A helpful example to keep in mind is $M = T^*X$.

- *Prequantization.* One assigns to the Poisson manifold M the space of L^2-functions on M, and functions become specific self-adjoint operators on

this space. This does not suffice as it "depends on too many variables" (as is the case with T^*X).

- *Quantization (Polarization).* In the second step, one chooses a nice Lagrangian foliation F of the Poisson manifold M and replaces $L^2(M)$ by $L^2(M/F)$. Integration along the leaves of the foliation relates the Hilbert spaces and the self-adjoint operators of the first and second step. (This step would replace T^*X by X.)
- *Metaplectic Correction.* While this process is mathematically sound, it does not always represent the physical process of quantization. In order to correct this, one has to do the metaplectic correction, which heavily depends on the concrete example at hand.

There is a different way to think about Poisson manifolds, because they happen to be in one-to-one correspondence with Lie algebroids. David's goal is to study and understand the geometric quantization process from this Lie algebroid perspective.

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Low-dimensional topology

Léo Bénard

Communicated by *Christian Bernert*

The greatest ideas of mathematicians often start simply with a pencil and a piece of paper. Suppose I put my pencil somewhere on this paper and start drawing a line. It doesn't have to be a straight line, it could be arbitrarily curved. We just ask that it should be *smooth*, so it should not have any corners or self-intersections. What can we end up with? Well, either we get bored after a while and just stop somewhere. Or maybe we feel a sense of harmony and return to the point where we started and stop there. Let us call these options Case A and Case B. Of course in each case, there are still infinitely many figures we could end up with, but from a rough point of view, they all look the same: All the figures one could have produced in Case A roughly look like a (deformed) line segment, while all the figures from Case B look like a loop, which, if you think about it, is just a deformed circle. So while all the figures in each one of the cases more or less look the same, a circle and a line segment are fundamentally different: For instance, the line segment has two points (the *boundary*) where it stops, while at every point on the circle you can go on in both directions. On the other hand, the

circle divides the piece of paper in two regions, one that is inside and one that is outside of the circle, while this does not happen for the line segment.

FIGURE 15.1. Typical outcomes of Case A (line) and Case B (circle)

Now Léo would say that what I just explained is a classification of *smooth, compact, connected one-dimensional manifolds*, namely each such object either looks like a circle or like a line segment. But he starts getting bored. Léo works in low-dimensional topology which means that he likes to study geometric objects that live in low dimensions, such as two, three or four dimensions. But one dimension is really too low, even for him. So let's move on.

Two-dimensional geometric objects are called *surfaces*. Again, we are interested in *smooth, compact, connected* surfaces, and we want to classify them up to *homeomorphism* which roughly speaking means that we identify objects that can be deformed into each other without changing too much of how they look.[15.1] As in the one-dimensional case, we also allow these surfaces to have some boundary. For example, a solid disk in the plane is a surface which has a circle as its boundary. On the other hand, a balloon (or a two-dimensional sphere) is a surface which has no boundary at all! However, there is more to surfaces than just the *boundary*: They can also have *holes*. For instance, the surface of a doughnut has one hole in the middle, while the surface of a pretzel has three and the balloon has no holes at all. If you ever started with a ball of dough and wanted to form a doughnut or a pretzel out of it, you will have noticed that it is not possible do to this without seriously changing the geometry of its surface at some point.

Léo is again getting bored. He tells me that the *genus* (which is just a slightly more fancy version of the number of holes) and the *number of boundary components* are called *topological invariants*. This just means that they are the same whenever two objects are homeomorphic. So it makes sense to use them for our classification of surfaces up to homeomorphism. But even better: These two invariants taken together are *complete*: We can check whether two surfaces are homeomorphic by just checking whether these two invariants agree on them. So we have completely solved the classification problem in the two-dimensional case. In fact, this was already done more than a century ago. So let's move on.

Léo is getting more excited. In three dimensions (3D), the fun only starts (or, if you are pessimist, the trouble). One initial problem is that our imagination starts to let us down with 3D manifolds. Just as we thought of surfaces as living in 3D space, it would be natural to think of 3D manifolds as living in 4D space. However this is quite hard for most human beings...

[15.1]This is not really the correct definition, but it is good enough for our purposes here.

FIGURE 15.2. A sphere and the surfaces of a doughnut and a pretzel

As we have already learned, if we want to solve the classification problem, we should try to find good and hopefully many *invariants*. For simplicity we restrict to manifolds without boundary. This leaves us with the idea of *the number of holes* from the case of surfaces. It is not immediately clear how to generalize this, but mathematicians have found a clever way to do it, by attaching to each manifold certain *homology groups* whose ranks we call *Betti Numbers* and these in some sense measure the number of holes in our manifold.

Unfortunately, this invariant is not very powerful: For instance, there are many very different manifolds which have the same Betti Numbers as the three-dimensional sphere S^3, but are not homeomorphic to it (for this reason, such manifolds are known as *(rational) homology spheres*).

There is a refinement of the first homology group, known as the *fundamental group* of M, which is denoted by $\pi_1(M)$. One can think of it as measuring how many different ways there are to put a loop inside our manifold. In the case of surfaces one can see that such a loop "gets stuck" at each hole, so one can imagine that this notion is just another way to capture the number of holes.[15.2]

As a bonus, the fundamental group is not only an abstract group, but it comes naturally with an action on another manifold $\widetilde{M}$, called the *universal cover* of M. Using the representation theory of $\pi_1(M)$, Reidemeister in the 1930's constructed a numerical invariant, the so-called *Reidemeister torsion* which allowed him to completely classify a certain subclass of 3D manifolds, the so-called *Lens spaces* – something that was not possible with previously known invariants!

Another important class of 3D manifolds arises as *knot complements*. Here, a *knot* is just a circle drawn in 3D space. As before, it is not allowed to intersect itself but of course this does not prevent the circle from knotting itself. If we think of the knot not just as a line but as a very thin solid tube, then the complement M of the knot forms a 3D manifold.[15.3]

Knots give an intriguing way to study 3D manifolds. On the one hand, they are very simple, combinatorial objects. On the other hand, they serve as "building blocks" for the theory: every 3D manifold can be obtained by gluing knot complements! Because of this combinatorial nature, one can find many invariants attached to knots. These *knot invariants* which could be numbers but also

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[15.2]It is the content of the Poincaré conjecture, the only one of the Millenium Problems solved so far, that the fundamental group can achieve what the homology groups could not: to distinguish the 3-sphere from 3D manifolds not homeomorphic to it

[15.3]Technically, since we are interested in compact manifolds, we need to consider it as a subset of $S^3 = \mathbb{R}^3 \cup \infty$ instead of just $\mathbb{R}^3$.

FIGURE 15.3. A 2D picture of a knot as an embedded circle in 3D space and its thickened version

more complicated algebraic objects, sometimes give interesting links between knot theory and other topics such as arithmetic and quantum physics.

Léo specifically is interested in knot invariants arising from a representation $\rho : \pi_1(M) \to G$ where G is a Lie group and M is the knot complement, such as the Reidemeister torsion. Here there are many interesting questions one can ask. For instance, fixing M and G, one can ask how the Reidemeister torsion behaves as one varies the representation ρ. One can also study the asymptotic behaviour of the Reidemeister torsion as M varies over a family. In many cases, this asymptotic behaviour recovers interesting geometric properties of our manifold M such as its volume.

Léo is finally happy. 3D manifolds are not going to get boring any time soon. There are many problems left to work on. I wish him good luck and a lot of fun!

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